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LETTER TO THE EDITOR

On the Mellin transforms of hypergeometric polynomials

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Abstract. Mellin integral transform pairs for all hypergeometric orthogonal polynomials in the Askey scheme, ranging from the classical Hermite polynomials up to the four-parameter Wilson polynomials, are systematically discussed.

1. Introduction

Integral transforms are an extensively used tool in solving various boundary value problems and integral equations. Recently, it has become clear that the classical integral transforms may also help us in revealing close relations between certain q-special functions. For example, various q-special functions exhibit simple behaviour with respect to the Fourier–Gauss integral transform although this transform is based on the q-independent exponential kernel. Instances of already known transforms of this type reveal novel relations between some polynomial families (such as the q-Hermite and q^{-1}-Hermite [1], the Rogers–Szegö and Stieltjes–Wigert [2], and the q-Laguerre and Wall polynomials [3]), q-exponential functions [4,5] and q-Bessel functions of Jackson [3].

To continue this line of research, we wish to discuss the Mellin integral transforms of hypergeometric orthogonal polynomials from the Askey scheme [6].

The Mellin integral transform is given by (see, for example, [7,8])

$$g(z) = \mathcal{M}\{f(t); z\} = \int_0^\infty f(t) t^{z-1}\,\mathrm{d}t. \tag{1.1}$$

As follows from the definition (1.1) the Mellin transforms of the functions $t^c f(t)$ and $\mathrm{d}^n f(t)/\mathrm{d}t^n$ are $g(z+c)$ and $(-1)^n(z-n)_n g(z-n)$, respectively, where $(z)_n = \Gamma(z+n)/\Gamma(z)$, $n = 0, 1, 2, \ldots,$ is the shifted factorial. Consequently,

$$\mathcal{M}\left\{t^k \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}; z\right\} = (-1)^n (z+k-n)_n g(z+k-n) \qquad k, n = 0, 1, 2, \ldots \tag{1.2}$$

and this means that if a function $f(t)$ satisfies some differential equation with polynomial coefficients in t, then its Mellin transform $g(z)$ is governed by a difference equation in the variable z, which is determined by the correspondence (1.2). Thus, (1.1) provides a general route for establishing integral transforms between solutions of differential and difference equations.

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A simple example of (1.1) is given by the Mellin transform of e^{-t} which is just the gamma function:

$$\Gamma(z) := \int_0^\infty \mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t \qquad \mathrm{Re}\, z > 0. \tag{1.3}$$

It is clear that the Mellin transform of e^{-t} times any polynomial in t gives a polynomial of the same degree in z, multiplied by the gamma function $\Gamma(z)$. Indeed, if $p_n(t) = \sum_{k=0}^n c_{n,k} t^k$ is some polynomial in the variable t, then

$$\int_0^\infty p_n(t)\mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t = \sum_{k=0}^n c_{n,k} \int_0^\infty \mathrm{e}^{-t}\, t^{z+k-1}\,\mathrm{d}t = \Gamma(z) \sum_{k=0}^n c_{n,k} (z)_k. \tag{1.4}$$

The inverse Mellin transform with respect to (1.3) is

$$\frac{1}{2\pi} \int_{-\infty}^\infty t^{-z}\Gamma(z)\,\mathrm{d}\mathrm{Im}\, z = \mathrm{e}^{-t} \qquad t > 0. \tag{1.5}$$

To verify (1.5), multiply both sides of (1.3) by τ^{-z}, $\tau > 0$, and integrate it with respect to Im z in infinite limits $(-\infty, \infty)$ by using the following property

$$\delta[\phi(x)] = \sum_s \frac{\delta(x - x_s)}{|\phi'(x_s)|} \qquad \phi'(x) = \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \tag{1.6}$$

of the δ-function, where x_s are zeros of the equation $\phi(x) = 0$.

Observe that the inverse Mellin transform of $\Gamma(z + k)$, where k is any positive integer number, yields $t^k\mathrm{e}^{-t}$ in complete agreement with the general correspondence rule (1.2) for the case when $n = 0$.

These properties of the Mellin transform (1.3) are known (see, for example, formulae (11) on p 337 and (27) on p 365 in [8]) and the relations (1.3)–(1.5) are sufficient to define the Mellin transform pairs for the hypergeometric orthogonal polynomials up to the third level in the Askey scheme [6]. Their explicit forms are discussed in sections 2–4.

It is more difficult to find the Mellin transform pairs for the family of dual Hahn polynomials (the fourth level) and the Wilson polynomials (the fifth level). In order to do so we need to know the Mellin transform pairs for the function $\Gamma(z)\Gamma(z^*) = |\Gamma(z)|^2$. With the aid of (1.3) and (1.6) it is not hard to evaluate that

$$\frac{1}{2\pi} \int_{-\infty}^\infty t^{-z}|\Gamma(z)|^2\,\mathrm{d}\mathrm{Im}\, z = \frac{\Gamma(2\mathrm{Re}\, z)}{(1+t)^{2\mathrm{Re}\, z}}. \tag{1.7}$$

Of course, the Mellin transform of the right-hand side of (1.7) reproduces $|\Gamma(z)|^2$. Indeed, to verify that

$$\Gamma(2\mathrm{Re}\, z) \int_0^\infty \frac{t^{z-1}\,\mathrm{d}t}{(1+t)^{2\mathrm{Re}\, z}} = \Gamma(2\mathrm{Re}\, z) B(z, z^*) = \Gamma(z)\Gamma(z^*) \tag{1.8}$$

one needs only to employ the well-known integral representation

$$B(z, \zeta) = \int_0^\infty \frac{t^{z-1}\,\mathrm{d}t}{(1+t)^{z+\zeta}} \qquad \mathrm{Re}\, z > 0, \quad \mathrm{Re}\, \zeta > 0 \tag{1.9}$$

for the beta function $B(z, \zeta)$ and the relation $B(z, \zeta) = \Gamma(z)\Gamma(\zeta)/\Gamma(z + \zeta)$.

The explicit forms (which are possibly new) of the Mellin transform pairs for the family of dual Hahn and the Wilson polynomials, obtained by using the relation (1.7), are discussed in sections 5 and 6, respectively.

This paper is thus aimed at assembling Mellin transform pairs $f(t)$ and $g(z)$ among all hypergeometric orthogonal polynomials from the Askey scheme [6], ranging from the classical Hermite polynomials up to the four-parameter Wilson polynomials.

2. The first level: the Hermite polynomials

The lowest level in the Askey scheme of hypergeometric orthogonal polynomials corresponds to the classical Hermite polynomials

$$H_n(x) := (2x)^n {}_2F_0(-n/2, -(n-1)/2; -1/x^2). \tag{2.1}$$

It is thus natural to start with the Mellin transform for $H_n(at)\mathrm{e}^{-t}$.

Depending on whether n is even or odd, the Hermite polynomials (2.1) contain only even or odd powers of x, respectively. Therefore, it is convenient to evaluate their Mellin transforms by using the formulae

$$H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} {}_1F_1\left(-n; \frac{1}{2}; x^2\right) = (-1)^n \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(\frac{1}{2})_k} \frac{x^{2k}}{k!} \tag{2.2a}$$

$$H_{2n+1}(x) = (-1)^n \frac{(2n+1)!}{n!} 2x \, {}_1F_1\left(-n; \frac{3}{2}; x^2\right) = (-1)^n \frac{(2n+1)!}{n!} 2x \sum_{k=0}^{n} \frac{(-n)_k}{(\frac{3}{2})_k} \frac{x^{2k}}{k!}. \tag{2.2b}$$

From (2.2*a*) it now follows that

$$\int_0^\infty t^{z-1} H_{2n}(at)\mathrm{e}^{-t}\,\mathrm{d}t = (-1)^n \frac{(2n)!}{n!} {}_3F_1\left(-n, z/2, (z+1)/2; \frac{1}{2}; 4a^2\right)\Gamma(z) \tag{2.3a}$$

upon employing the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z + \frac{1}{2}\right) \tag{2.4}$$

for the gamma function $\Gamma(z)$. Similarly, from (2.2*b*) one obtains that

$$\int_0^\infty t^{z-1} H_{2n+1}(at)\mathrm{e}^{-t}\,\mathrm{d}t = (-1)^n \frac{(2n+1)!}{n!} {}_3F_1\left(-n, 1+z/2, (z+1)/2; \frac{3}{2}; 4a^2\right)\Gamma(z). \tag{2.3b}$$

Thus, the Mellin transform of $H_n(at)\mathrm{e}^{-t}$ has the form

$$\int_0^\infty t^{z-1} H_n(at)\mathrm{e}^{-t}\,\mathrm{d}t = g_n(z; 2a)\Gamma(z) \tag{2.5}$$

where

$$g_{2n}(z; a) = (-1)^n \frac{(2n)!}{n!} {}_3F_1\left(-n, z/2, (z+1)/2; \frac{1}{2}; a^2\right) \tag{2.6a}$$

$$g_{2n+1}(z; a) = (-1)^n \frac{(2n+1)!}{n!} az \, {}_3F_1\left(-n, 1+z/2, (z+1)/2; \frac{3}{2}; a^2\right). \tag{2.6b}$$

Observe that $g_n(z; a)$ are polynomials of nth degree in both the variable z and the parameter a.

Since the Hermite polynomials (2.1) satisfy a second-order differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + 2n\right] H_n(x) = 0 \tag{2.7}$$

from the correspondence rule (1.2) it follows that the polynomials $g_n(z; a)$ are solutions of the difference equation

$$\left\{2\left[1 - \exp\left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)\right]^2 + a^2 z\left[1 - \exp\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\right] + na^2\right\} g_n(z; a) = 0 \tag{2.8}$$

with respect to the variable z.

3. The second level: the Laguerre and the Charlier polynomials

The second level in the Askey scheme of hypergeometric orthogonal polynomials corresponds to the Laguerre and the Charlier polynomials, which depend on one parameter.

The Laguerre polynomials. We start with a Mellin transform pair for the Laguerre polynomials, which are defined as

$$L_n^{(\alpha)}(z) := \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;z) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{z^k}{k!}. \tag{3.1}$$

The Laguerre polynomials (3.1) satisfy a second-order differential equation

$$\left[z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (\alpha+1-z)\frac{\mathrm{d}}{\mathrm{d}z} + n\right]L_n^{(\alpha)}(z) = 0. \tag{3.2}$$

The Mellin transform of $L_n^{(\alpha)}(at)\mathrm{e}^{-t}$ is

$$\int_0^\infty L_n^{(\alpha)}(at)t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t = \frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n,z;\alpha+1;a)\Gamma(z). \tag{3.3}$$

This follows from the definition (3.1) upon employing the integral representation (1.3) for the gamma function $\Gamma(z)$.

The hypergeometric polynomials ${}_2F_1$ in the right-hand side of (3.3) are in fact the Meixner–Pollaczek polynomials

$$\begin{aligned} P_n^{(\nu)}(y;\phi) &:= \frac{(2\nu)_n}{n!}\mathrm{e}^{\mathrm{i}n\phi}\,{}_2F_1(-n,\nu+\mathrm{i}y;2\nu;1-\mathrm{e}^{-2\mathrm{i}\phi}) \\ &= \frac{(2\nu)_n}{n!}\mathrm{e}^{\mathrm{i}n\phi}\sum_{k=0}^{n}\frac{(-n)_k(\nu+\mathrm{i}y)_k}{(2\nu)_k k!}(1-\mathrm{e}^{-2\mathrm{i}\phi})^k \end{aligned} \tag{3.4}$$

which depend on two parameters ν and ϕ and therefore correspond to the third level in the Askey scheme [6].

Thus from (3.3) and (3.4) we obtain that

$$\int_0^\infty L_n^{(2\mathrm{Re}\,z-1)}((1-\mathrm{e}^{-2\mathrm{i}\phi})t)t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t = \mathrm{e}^{-\mathrm{i}n\phi}P_n^{(\mathrm{Re}\,z)}(\mathrm{Im}\,z;\phi)\Gamma(z). \tag{3.5}$$

The inverse Mellin transform with respect to (3.5) is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z}P_n^{(\mathrm{Re}\,z)}(\mathrm{Im}\,z;\phi)\Gamma(z)\,\mathrm{dIm}\,z = \mathrm{e}^{\mathrm{i}n\phi}L_n^{(2\mathrm{Re}\,z-1)}((1-\mathrm{e}^{-2\mathrm{i}\phi})t)\mathrm{e}^{-t}. \tag{3.6}$$

As a consistency check one can combine (3.5) and (3.6) with generating functions for the Laguerre and Meixner–Pollaczek polynomials to obtain a Mellin transform for the Bessel function $J_\alpha(z)$. Indeed, these generating functions are of the form

$$\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}(x) = (xt)^{-\alpha/2}\mathrm{e}^t J_\alpha\left(2\sqrt{xt}\right) \qquad \alpha > -1 \tag{3.7}$$

and

$$\sum_{n=0}^{\infty}\frac{(t\mathrm{e}^{-\mathrm{i}\phi})^n}{(2\nu)_n}P_n^{(\nu)}(x;\phi) = \mathrm{e}^t\,{}_1F_1[\nu+\mathrm{i}x;2\nu;(\mathrm{e}^{-2\mathrm{i}\phi}-1)t]. \tag{3.8}$$

Multiplying both sides of (3.5) by $x^n\mathrm{e}^{\mathrm{i}n(\phi-\pi/2)}/2^n(2\mathrm{Re}\,z)_n$ and summing over n from zero to infinity with the aid of (3.7) and (3.8), yields an integral transform

$$\int_0^\infty t^{\mathrm{i}y-1/2}J_{2\nu-1}(2\sqrt{xt})\mathrm{e}^{-t}\,\mathrm{d}t = \frac{\Gamma(\nu+\mathrm{i}y)}{\Gamma(2\nu)}x^{\nu-1/2}\mathrm{e}^{-x}\,{}_1F_1(\nu-\mathrm{i}y;2\nu;x) \tag{3.9}$$

where $\nu = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

In a similar manner, from (3.6) follows the inverse integral transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-\mathrm{i}y}\Gamma(\nu+\mathrm{i}y)\,{}_1F_1(\nu-\mathrm{i}y;2\nu;x)\,\mathrm{d}y = \Gamma(2\nu)x^{1/2-\nu}t^{1/2}\mathrm{e}^{x-t}J_{2\nu-1}\left(2\sqrt{xt}\right). \qquad (3.10)$$

The change of the variables $t^{1/2} \to t$, $x^{1/2} \to x$, and the shift of the parameter $\nu \to \nu + \frac{1}{2}$ in (3.9) lead to an integral transform

$$\int_0^{\infty} t^{2\mathrm{i}y}J_{2\nu}(2xt)\mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{\Gamma(\nu+\mathrm{i}y+\frac{1}{2})}{2\Gamma(2\nu+1)}x^{2\nu}\mathrm{e}^{-x^2}\,{}_1F_1(\nu-\mathrm{i}y+\tfrac{1}{2};2\nu+1;x^2) \qquad (3.11)$$

that is contained in [8, 9].

We close this subsection by the following remark. The Meixner–Pollaczek polynomials (3.4) are known to be solutions of the difference equation

$$\left\{\mathrm{e}^{\mathrm{i}\phi}(\nu-\mathrm{i}x)\exp\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right) - \mathrm{e}^{-\mathrm{i}\phi}(\nu+\mathrm{i}x)\exp\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right) + 2\mathrm{i}[x\cos\phi-(n+\nu)\sin\phi]\right\}$$
$$\times P_n^{(\nu)}(x;\phi) = 0. \qquad (3.12)$$

One may verify that the differential equation (3.2) for the Laguerre polynomials (3.1) and the difference equation (3.12) are thus interrelated by the Mellin transforms (3.5) and (3.6) in complete agreement with the correspondence rule (1.2).

The Charlier polynomials. The Charlier polynomials $C_n(x;a)$ are defined as

$$C_n(x;a) := {}_2F_0(-n,-x;-1/a) = \sum_{k=0}^{n}(-1)^k\frac{(-n)_k(-x)_k}{k!a^k}. \qquad (3.13)$$

They satisfy the difference equation

$$\left[a\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) + x\exp\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right) + n - x - a\right]C_n(x;a) = 0. \qquad (3.14)$$

From (1.4) and (3.13) follows that the inverse Mellin transform of $C_n(-z;a)\Gamma(z)$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z}C_n(-z;a)\Gamma(z)\,\mathrm{dIm}\,z = (1+t/a)^n\mathrm{e}^{-t} \qquad (3.15)$$

where we have used the relation $(-n)_k = (-1)^k\binom{n}{k}k!$. In other words, $C_n(-z;a)\Gamma(z)$ is the Mellin transform of the simple expression $(1+t/a)^n\mathrm{e}^{-t}$. According to the correspondence rule (1.2), the difference equation (3.14) and the inverse Mellin transform (3.15) lead to the first-order differential equation

$$\left[(t+a)\frac{\mathrm{d}}{\mathrm{d}t} - n\right](1+t/a)^n = 0 \qquad (3.16)$$

satisfied by the nth power of the binomial $1+t/a$.

4. The third level: the Jacobi and the Meixner polynomials

The third level in the Askey scheme of hypergeometric orthogonal polynomials [6] corresponds to the Meixner–Pollaczek, Jacobi, Meixner and Kravchuk polynomials, which depend on two parameters. As we have shown in the previous section, the Meixner–Pollaczek polynomials are related by the Mellin transform with the Laguerre polynomials.

The Jacobi polynomials. A two-parameter family of the Jacobi polynomials is defined by

$$P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right)$$
$$= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{2^k k!(\alpha+1)_k}(1-z)^k. \tag{4.1}$$

These polynomials satisfy a second-order differential equation of the form

$$\left\{(1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + [\beta-\alpha-(\alpha+\beta+2)z]\frac{\mathrm{d}}{\mathrm{d}z} + n(n+\alpha+\beta+1)\right\} P_n^{(\alpha,\beta)}(z) = 0. \tag{4.2}$$

Using the definition (4.1) and the integral representation (1.3), it is not hard to evaluate that

$$\int_0^\infty P_n^{(\alpha,\beta)}(1-2xt)t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t = \frac{(\alpha+1)_n}{n!}\,{}_3F_1(-n, z, n+\alpha+\beta+1; \alpha+1; x)\Gamma(z). \tag{4.3}$$

The inverse Mellin transform of (4.3) is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z}{}_3F_1(-n, z, n+\alpha+\beta+1; \alpha+1; x)\Gamma(z)\,\mathrm{dIm}\,z = \frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(1-2xt)\mathrm{e}^{-t}. \tag{4.4}$$

Thus, as in the case of the Laguerre polynomials, the Mellin transforms (4.3) and (4.4) connect the third level in the hierarchical ladder of hypergeometric orthogonal polynomials [6] with the next one.

Note that the Jacobi polynomials (4.1) with particular values of the parameters α and β are known to correspond to: the Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(x)$, when $\alpha = \beta = \lambda - \frac{1}{2}$; the Chebyshev polynomials of the first $T_n(x)$ and the second $U_n(x)$ kind, if $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, respectively; and the Legendre (or spherical) polynomials $P_n(x)$, when $\alpha = \beta = 0$. The Mellin transforms (4.3) and (4.4) therefore contain all these special cases of the Jacobi polynomials.

The Meixner and the Kravchuk polynomials. The Meixner polynomials $M_n(z; \beta, c)$ are defined by

$$M_n(z; \beta, c) = {}_2F_1(-z, -n; \beta; 1-1/c). \tag{4.5}$$

Since they satisfy the difference equation

$$\left[c(z+\beta)\exp\left(\frac{\mathrm{d}}{\mathrm{d}z}\right) + z\exp\left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)\right]M_n(z;\beta,c) = [(c+1)z + n(c-1) + \beta c]M_n(z;\beta,c) \tag{4.6}$$

we need to evaluate the inverse Mellin transform of the $M_n(-z; \beta, c)\Gamma(z)$ (cf (3.15)). From (1.4) and (4.5) it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z}M_n(-z;\beta,c)\Gamma(z)\,\mathrm{dIm}\,z = {}_1F_1(-n; \beta; (1-1/c)t)\mathrm{e}^{-t}$$
$$= \frac{n!}{(\beta)_n}L_n^{(\beta-1)}((1-1/c)t)\mathrm{e}^{-t}. \tag{4.7}$$

It remains only to remind the reader that the Kravchuk polynomials

$$K_n(z; p, N) := {}_2F_1(-z, -n; -N; 1/p) \qquad n = 0, 1, \ldots, N \tag{4.8}$$

can be obtained from the Meixner polynomials (4.5) by substituting $\beta = -N$ and $c = p/(p-1)$.

5. The fourth level: the continuous Hahn and the dual Hahn polynomials

The fourth level in the Askey scheme of hypergeometric orthogonal polynomials is occupied by the continuous Hahn, the Hahn, the continuous dual Hahn and the dual Hahn polynomials.

The continuous Hahn and the Hahn polynomials. The evaluation of the inverse Mellin transforms for the continuous Hahn polynomials [6, 10]

$$\begin{aligned} h_n(z;a,b,c,d) &:= \frac{\mathrm{i}^n}{n!}(a+c)_n(a+d)_n\, {}_3F_2\left(\left.\begin{matrix}-n, n+a+b+c+d-1, a+\mathrm{i}z\\ a+c, a+d\end{matrix}\right|1\right)\\ &= \frac{\mathrm{i}^n}{n!}(a+c)_n(a+d)_n\sum_{k=0}^{n}\frac{(-n)_k(n+a+b+c+d-1)_k}{k!(a+c)_k(a+d)_k}(a+\mathrm{i}z)_k \end{aligned} \tag{5.1}$$

and the Hahn polynomials

$$Q_n(z;\alpha,\beta,N) := {}_3F_2\left(\left.\begin{matrix}-n, n+\alpha+\beta+1, -z\\ \alpha+1,\ -N\end{matrix}\right|1\right) \tag{5.2}$$

can be easily performed in exactly the same way as for the polynomials from the previous two levels. The result is

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-(a+\mathrm{i}y)}h_n(y;a,b,c,d)\Gamma(a+\mathrm{i}y)\,\mathrm{d}y\\ &\qquad = \frac{\mathrm{i}^n}{n!}(a+c)_n(a+d)_n\, {}_2F_2\left(\left.\begin{matrix}-n, n+a+b+c+d-1\\ a+c, a+d\end{matrix}\right|t\right)\mathrm{e}^{-t} \end{aligned} \tag{5.3}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z}Q_n(-z;\alpha,\beta,N)\Gamma(z)\,\mathrm{d}\mathrm{Im}\, z = {}_2F_2\left(\left.\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1, -N\end{matrix}\right|t\right)\mathrm{e}^{-t} \tag{5.4}$$

respectively.

The correspondence rule (1.2) transforms the difference equation

$$\begin{aligned} &[(a+\mathrm{i}z)(b+\mathrm{i}z)(\mathrm{e}^{-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}}-1)+(c-\mathrm{i}z)(d-\mathrm{i}z)(\mathrm{e}^{\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}}-1)]h_n(z;a,b,c,d)\\ &\qquad = n(n+a+b+c+d-1)h_n(z;a,b,c,d) \end{aligned} \tag{5.5}$$

for the continuous Hahn polynomials (5.1) into a third-order differential equation

$$\begin{aligned} &\left\{t^2\frac{\mathrm{d}^3}{\mathrm{d}t^3}+[2a+c+d+1-t]t\frac{\mathrm{d}^2}{\mathrm{d}t^2}+[(a+c)(a+d)\right.\\ &\qquad\left.-(a+b+c+d)t]\frac{\mathrm{d}}{\mathrm{d}t}+n(n+a+b+c+d-1)\right\}\\ &\qquad\times {}_2F_2\left(\left.\begin{matrix}-n,\ n+a+b+c+d-1\\ a+c, a+d\end{matrix}\right|t\right)=0 \end{aligned} \tag{5.6}$$

in the variable t. The ${}_2F_2$-polynomials in (5.4) satisfy the same type of differential equation, which is readily obtained from (5.6) by the substitutions $a+c\to\alpha+1$, $a+d\to -N$, and $a+b+c+d\to\alpha+\beta+2$.

The continuous dual Hahn and the dual Hahn polynomials. The situation with the two remaining families at the fourth level is a little different. The point is that the continuous dual Hahn and the dual Hahn polynomials are defined [6] as

$$S_n(x^2;a,b,c) = (a+b)_n(a+c)_n\, {}_3F_2\left(\left.\begin{matrix}-n, a+\mathrm{i}x, a-\mathrm{i}x\\ a+b, a+c\end{matrix}\right|1\right) \tag{5.7}$$

and

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\begin{matrix}-n,-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{matrix}\middle|1\right) \qquad n=0,1,2,\ldots,N$$
$$\lambda(x) = x(x+\gamma+\delta+1) \tag{5.8}$$

respectively. In both of these cases the dependence on the variable enters the product of two shifted factorials $(a+\mathrm{i}x)_k$ and $(a-\mathrm{i}x)_k$, or $(-x)_k$ and $(x+\gamma+\delta+1)_k$, respectively. To evaluate the inverse Mellin transforms of (5.7), or (5.8), it is therefore necessary to use the relation (1.7), rather than (1.5). Indeed, from (5.7) and (1.7) with the aid of the duplication formula (2.4) one obtains

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z} S_n((\operatorname{Im} z)^2;a,b,c)|\Gamma(z)|^2\,\mathrm{dIm}\,z = (a+b)_n(a+c)_n\frac{\Gamma(2a)}{(1+t)^{2a}}$$
$$\times{}_3F_2\left(\begin{matrix}-n,a,a+\frac{1}{2}\\ a+b,a+c\end{matrix}\middle|\frac{4t}{(1+t)^2}\right) \qquad \operatorname{Re} z = a > 0. \tag{5.9}$$

Similarly, from (5.8) and (1.7) follows that the inverse Mellin transform of the dual Hahn polynomials is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z} R_n(-|z|^2;\gamma,\delta,N)|\Gamma(z)|^2\,\mathrm{dIm}\,z$$
$$= \frac{\Gamma(2\operatorname{Re} z)}{(1+t)^{2\operatorname{Re} z}}{}_3F_2\left(\begin{matrix}-n,\operatorname{Re} z,\operatorname{Re} z+\frac{1}{2}\\ \gamma+1,-N\end{matrix}\middle|\frac{4t}{(1+t)^2}\right) \tag{5.10}$$
$$\operatorname{Im}(\gamma+\delta) = 0 \qquad 2\operatorname{Re} z = \gamma+\delta+1 > 0.$$

6. The fifth level: the Wilson and the Racah polynomials

The top level in the Askey scheme of hypergeometric orthogonal polynomials is occupied by the Wilson and the Racah polynomials, which are defined [6] as

$$W_n(x^2;a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n$$
$$\times\,{}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+\mathrm{i}x,a-\mathrm{i}x\\ a+b,a+c,a+d\end{matrix}\middle|1\right) \tag{6.1}$$

and

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix}\middle|1\right) \tag{6.2}$$
$$n=0,1,2,\ldots,N \qquad \lambda(x)=x(x+\gamma+\delta+1)$$

respectively.

From the relations (6.1) and (1.7) with the aid of the duplication formula (2.4) one obtains the inverse Mellin transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-(a+\mathrm{i}y)} W_n(y^2;a,b,c,d)|\Gamma(a+\mathrm{i}y)|^2\,\mathrm{d}y = (a+b)_n(a+c)_n(a+d)_n\frac{\Gamma(2a)}{(1+t)^{2a}}$$
$$\times{}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a,a+\frac{1}{2}\\ a+b,a+c,a+d\end{matrix}\middle|\frac{4t}{(1+t)^2}\right) \tag{6.3}$$

for the Wilson polynomials ($\operatorname{Re} a > 0$).

Similarly, from (6.2) and (1.7) follows that the inverse Mellin transform for the Racah polynomials has the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-z} R_n(-|z|^2;\alpha,\beta,\gamma,N)|\Gamma(z)|^2\,\mathrm{dIm}\,z$$
$$= \frac{\Gamma(2\mathrm{Re}\,z)}{(1+t)^{2\mathrm{Re}\,z}}$$
$${}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, \mathrm{Re}\,z, \mathrm{Re}\,z+\frac{1}{2}\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}\,\middle|\,\frac{4t}{(1+t)^2}\right)$$

$2\mathrm{Re}\,z = \gamma+\delta+1 > 0.$

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